EXAMPLES OF NON-FORMAL CLOSED $(k-1)$-CONNECTED MANIFOLDS OF DIMENSIONS $4k-1$ AND MORE

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Abstract. We construct closed $(k-1)$-connected manifolds of dimensions $\geq 4k-1$ that possess non-trivial rational Massey triple products. We also construct examples of manifolds M such that all the cup-products of elements of $H^k(M)$ vanish, while the group $H^{3k-1}(M;\mathbb{Q})$ is generated by Massey products: such examples are useful for theory of systols.

For every k we construct closed $(k-1)$-connected manifolds of dimensions $\geq 4k-1$ that possess non-trivial rational Massey triple products and therefore are non-formal. For $k=1$ such manifolds can be obtained as the products of Heisenberg manifold with circles. For $k=2$ such examples are also known, see e.g. [4, 2], but even in this case our construction seems more direct and simple.

Miller [3] proved that every closed $(k-1)$-connected manifold M of dimension $\leq 4k-2$ is formal. In particular, all rational Massey products in M vanish. So, neither Miller's nor our results can be improved.

Given a diagram

$$B \supset A \xrightarrow{\ f\ } Y$$

we denote by Z_f its double mapping cylinder.

Recall that a subset S of a space $\mathbb{R}^m$ is called *radial* if, for all points $s \in S$, the linear segment $[0,s]$ contains precisely one point of S (namely, s).

1. **Proposition.** *Let B be a finite polyhedron in $\mathbb{R}^m, m>1$, let A be a subpolyhedron of B such that $A \setminus \{0\}$ is radial in $\mathbb{R}^m$, and let Y be a finite polyhedron in $\mathbb{R}^n$. Then the double cylinder Z_f of any simplicial map $f: A \to Y$ admits a PL embedding in $\mathbb{R}^{m+n}$.*

Proof. We denote by 0_m and 0_n the origins of spaces R^m and R^n, respectively. We first consider the case when $0_m \notin A$, We assume that Y is far away from 0_n. Let $\Gamma \in \mathbb{R}^m \times \mathbb{R}^n$ be the graph of the map f. We join every point $(x, f(x)) \in \Gamma, x \in A$ with the point $(0_m, f(x)) \in \mathbb{R}^m \times Y \subset \mathbb{R}^{m+n}$ by the linear segment. Then, since A is radial, we get an embedding of the mapping cylinder M_f of f to $\mathbb{R}^{m+n}$. Moreover, if we join the points $(x, 0_n)$ with $(x, f(x))$ by the linear segment, we still have an embedding $M_f \hookrightarrow \mathbb{R}^{m+n}$. Here (the image of) M_f is formed by segments $[(x,0_n),(x,f(x))]$ and $[(x,f(x)),(0_m,f(x))]$. Finally, we get an embedding of the double mapping cylinder Z_f to $\mathbb{R}^{m+n}$ by adding the space B to the embedded mapping cylinder M_f.

The case $0_m \in A$ can be considered similarly. We can assume that there is a point $y_0 \in Y$ which is the closest to $0_n \in \mathbb{R}^n$, i.e. $||y_0|| < ||y||$ if $y \neq y_0$ and $y \in Y$. We can also assume that $f(0_m) = y_0$. Consider the map $f' = f|(A \setminus \{0\})$ and the

embedding $i : Z_{f'} \to \mathbb{R}^{m+n}$ as above. Then $i(Z_{f'}) \cup [0_m, y_0]$ is an embedding of Z_f. □

2. **Corollary.** *Let Y be a finite polyhedron in $\mathbb{R}^n$, and let $f : \vee_i S_i^{m-1} \to Y, i = 1, \dots, k$ be a simplicial map, where S_i^{m-1} is the copy of the sphere S^{m-1}. Then the cone C_f of f can be simplicially embedded in $\mathbb{R}^{m+n}$.*

Proof. Choose a base point on the boundary of each disc $D_i^m, i = 1, \dots, k$ and consider the wedge $\vee_{i=1}^m D_i^m$. We can regard this wedge as a polyhedron in $\mathbb{R}^m$ such that the base point is the origin and $\vee S_i^{m-1} \setminus \{0\}$ is a radial set. Now the claim follows from Proposition 1. □

Consider the wedge $K = S^{k_1} \vee S^{k_2} \vee S^{k_3}$ of spheres with $k_i \geq 2$ and let $\iota_r \in \pi_{k_r}(K)$ be represented by the inclusion map $S^{k_r} \subset K$. Set $m = k_1 + k_2 + k_3 - 1$, let $f : S^{m-1} \to K$ represent the element $[\iota_1, [\iota_2, \iota_3]]$, and let X be the cone of the map f. Let $\alpha_i \in H^{k_i}(X)$ be the cohomology class which takes the value 1 on the cell S_i^k of X and 0 on other cells. We recall the following classical result

3. **Theorem.** *The Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{k_1+k_2+k_3-1}(X)$ has the zero indeterminacy and takes the value $(-1)^{k_1}$ on the $(m-1)$-dimensional cell of X.*

Proof. See [5, Lemma 7]. □

Now let $k_1 = k_2 = k_3 = k$ and consider the corresponding space X. According to Proposition 1, X admits a PL embedding in $\mathbb{R}^N$ with $N \geq 4k$. Fix such an embedding and let W be a closed regular neighborhood of X in $\mathbb{R}^N$. So, W is a manifold with the boundary $V = \partial W$. Furthermore, W has the homotopy type of X. (Notice that W is a PL manifold by the construction, but without loss of generality we can assume that W is smooth.)

4. **Proposition.** *The manifold V is $(k-1)$-connected.*

Proof. Consier a sphere $S^i, i < k$ in V. Since W is $(k-1)$-connected, there exists a disk D^{i+1} in W with $\partial D^{i+1} = S^i$. Since $i + 1 + \dim X \leq 4k - 1 < N$, we can assume that $D^{i+1} \cap X = \emptyset$. But V is a retract of $W \setminus X$, and thus S^i bounds a disk in V. □

5. **Proposition.** $H^i(W, V) = H_{N-i}(X)$.

Proof. We have
$$H^i(W, V) = H_{N-i}(W) = H_{N-i}(X)$$
where the first equality holds by the Poincaré duality, see e.g. Dold [1]. □

Consider the map
$$g : V \xrightarrow{\ i\ } W \xrightarrow{\ r\ } X$$
where i is the inclusion and r is a deformation retraction.

6. **Theorem.** *If $N \neq 5k-1, 6k-2$, then the Massey product $\langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3 \rangle$ has zero indeterminacy and is non-zero.*

Proof. Notice that $H_i(X) = 0$ for $i \neq 0, k, 3k-1$. We have $H^{2k-1}(W) = H^{2k-1}(X) = 0$ and $H^{2k}(W, V) = H_{n-2k}(X) = 0$. Now, in view of the exactness of the sequence $H^{2k-1}(W) \to H^{2k-1}(V) \to H^{2k}(W, V)$ we have $H^{2k-1}(V) = 0$, and therefore the indeterminacy of the Massey product is zero. Furthermore, the map $i^* : H^{3k-1}(W) \to H^{3k-1}(V)$ is injective since $H^{3k-1}(W, V) = H_{n-3k+1}(X) = 0$.

Thus, the map $g^*: H^{3k-1}(X) \to H^{3k-1}(V)$ is injective. But $g^*\langle \alpha_1, g_*\alpha_2, g^*\alpha_3\rangle = \langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3\rangle$ because both parts of the equality have zero indeteminaces. $\square$

Thus, we have examples of $(k-1)$-connected manifolds with non-trivial triple Massey product of dimensions $d \geq 4k-1$ but $d \neq 5k-2$, $6k-3$. In order to construct an example in exceptional dimensions, just take the double of the manifold W (or multiple by the sphere of the correspondent dimension if $k \neq 2$).

When we put the first version of the paper into the e-archive, Mikhail Katz asked us if we can construct a closed manifold M such that all the cup-products of elements of $H^k(M)$ vanish, while the group $H^{3k-1}(M;\mathbb{Q})$ is generated by Massey products. Now we present such an example.

7. **Lemma.** *Consider a wedge $X \vee Y$ and three elements $u, v, w \in H^*(X)$ such that $uv = 0$, $u|Y = 0 = v|Y$ and $w|X = 0$. Then all the Massey products $\langle u, v, w\rangle$, $\langle u, w, v\rangle$ and $\langle w, u, v\rangle$ are trivial, i.e. they contain the zero element.*

Proof. This follows from the following fact: If $A \in C^*(X \vee Y)$ and $B \in C^*(X \vee Y)$ are cochains with the supports in X and Y, respectively, than their product is equal to zero. We leave the details to the reader. $\square$

Consider the wedge $S^k_1 \vee S^k_2 \vee S^k_3 \vee S^k_4$ of k-dimensional spheres, $k > 1$. Let $\iota_m \in \pi_k(S^k_m)$ be the generator. Set

$$Z = \left(\vee_{i=1}^4 S^k_i\right) \cup_{f_1} e^{3k-1} \tag{1}$$

where $f_1 : S^{3k-2} \to \vee_{i=1}^4 S^k_i$ represents the homotopy class $[\iota_1, [\iota_2, \iota_3]]$. Let $\alpha_i \in H^k(Z)$ be the cohomology class which takes the value 1 on the cell S^k_i of Z and 0 on other cells.

8. **Corollary.** *If at least one of the indices i, j, k is equal to 4, then $\langle \alpha_i, \alpha_j, \alpha_k\rangle = 0$ in X.*

Proof. This follows directly from Lemma 7 since

$$Z = \left(\left(\vee_{i=1}^3 S^k_i\right) \cup_{f_1} e^{3k-1}\right) \vee S^k_4.$$

$\square$

For convenience of notation, we set $\iota_5 = \iota_1$ and $\iota_6 = \iota_2$. Let $f_m : S^{3k-2} \to \vee_{i=1}^4 S^k_i$ be the map which represents $[\iota_m, [\iota_{m+1}, \iota_{m+2}]], m = 1, 2, 3, 4$. Consider the map

$$f : \vee_{i=1}^4 S^{3k-2}_i \to \vee_{i=1}^4 S^k_i$$

such that $f|S^{3k-2}_i = f_i$ and set $X = C_f$. We define $\alpha_m \in H^k(X)$ the cohomology class which takes the value 1 on the cell S^k_i of X and 0 on other cells. For convenience of notation, we set $\alpha_5 = \alpha_1$ and $\alpha_6 = \alpha_2$.

9. **Lemma.** *The homology classes $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2}\rangle$ are linearly independent in $H^{3k-1}(X)$.*

Proof. First, notice all these Massey products are defined and have zero indeterminacies. Now, suppose that $\sum_{m=1}^4 c_m\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2}\rangle = 0$ for some $c_m \in \mathbb{R}$. Consider the space Z as in (1) and the obvious inclusion $j : Z \to X$. Then $j^*\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2}\rangle = 0$ for $m = 2, 3, 4$ by Corollary 8, while $j^*\langle \alpha_1, \alpha_2, \alpha_3\rangle \neq 0$ by Theorem 3. Therefore $c_1 = 0$. Similarly, we can prove that $c_m = 0$ for all m. $\square$

Now, because of Proposition 1, X can be regarded as a polyhedron in $\mathbb{R}^N$ with $N \geq 4k$. Let W be a regular neighborhood of X in $\mathbb{R}^N$ and set $M = \partial W$.

10**. Theorem.** *If $N \neq 4k$, $5k-1$, $6k-2$, $6k-1$ then $H^{3k-1}(M;\mathbb{Q})$ is generated by triple Massey products, while all the cup-products of elements of $H^k(M)$ vanish.*

Proof. Consider the map

$$g : V \xrightarrow{\ i\ } W \xrightarrow{\ r\ } X$$

where i is the inclusion and r is a deformation retraction. Using the isomorphisms $H^i(W,M) \cong H_{N-i}(X)$ and $H^i(W) \cong H^i(X)$, and the exactness of the sequence

$$H^i(W,M) \longrightarrow H^i(W) \xrightarrow{\ i^*\ } H^i(M) \longrightarrow H^{i+1}(W,M).$$

we conclude that $H^{2k-1}(M) = 0$ and

$$g^* : H^{3k-1}(X) \to H^{3k-1}(M)$$

is an isomorphism. Now, the equality $H^{2k-1}(M) = 0$ implies that all the Massey products $\langle \alpha_i, \alpha_j, \alpha_k \rangle$ have zero indeterminacies. Furthermore, since g^* is an isomorphism, Lemma 9 implies that the g^*-images of the classes $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle$, $m = 1, 2, 3, 4$ in M form a basis of $H^{3k-1}(M;\mathbb{Q})$. Finally, the map $i^* : H^k(W) \to H^k(M)$ is surjective for $N \neq 4k$, and so the cup-products of elements of $H^k(M)$ vanish. □

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